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THE CONTINUOUS DEPENDENCE ON φ OF SOLUTIONS OF $u_t - \Delta\varphi(u) = 0$

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ABSTRACT

The initial-value problem for equations of the form $u_t - \Delta\varphi(u) = 0$ where $\varphi: \mathbb{R} \to \mathbb{R}$ is nondecreasing arises in many contexts. The main results of this paper concern the continuity of the solutions of this initial-value problem as a function of φ. Depending on the behavior of φ near zero, one finds either that the solutions are continuous into $C([0,\infty):L^1(\mathbb{R}^N))$ as a function of φ or into a weaker space in which $L^1(\mathbb{R}^N)$ is replaced by a certain weighted L^1 space. A variety of auxiliary results are proved and the sharpness of the condition which distinguishes between the above cases is established.

AMS(MOS) Subject Classification: 35K55, 35K15, 47H15

Key Words: quasilinear parabolic equations
m-accretive operator,
continuous dependence,
porous flow problems.

Work Unit #1 - Applied Analysis

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Significance and Explanation

The initial-value problem for equations of the form $u_t - \Delta\varphi(u) = 0$ where $\varphi: \mathbb{R} \to \mathbb{R}$ is nondecreasing arises in many contexts. The main results of this paper concern the continuity of the solutions of this initial-value problem as a function of φ. This question is of interest from many points of view. To have a physically meaningful problem, one wants continuity in φ. Or, for example, one might like to approximate solutions of the problem $u_t - \Delta(u^3) = 0$ (which is degenerate where the solution vanishes) by solutions of the nondegenerate problem $u_t - \Delta(u^3 + \varepsilon u) = 0$ for $\varepsilon > 0$. This is justified by the current work.

THE CONTINUOUS DEPENDENCE ON φ OF SOLUTIONS OF $u_t - \Delta\varphi(u) = 0$

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Introduction.

The initial-value problem for equations of the form $u_t - \Delta\varphi(u) = 0$, where $\varphi: \mathbb{R} \to \mathbb{R}$ is nondecreasing, arises in many contexts. The main results of this paper concern the continuity of solutions of these initial-value problems as functions of the nonlinearity φ . These results will be obtained via nonlinear semigroup theory, in a generality in which φ may be a monotone graph; however we will preview the results here in a more restrictive setting.

If $\varphi: \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing and $u_0 \in L^1(\mathbb{R}^\infty) \cap L^\infty(\mathbb{R}^N)$, then there is a unique $u \in C([0,\infty) : L^1(\mathbb{R}^N)) \cap L^\infty((0,\infty) \times \mathbb{R}^N)$ which satisfies

$$(1) \quad \begin{cases} u_t - \Delta\varphi(u) = 0 & \text{in} \quad \mathcal{D}'((0,\infty) \times \mathbb{R}^N), \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

The existence assertion is explained in Section 1 and the uniqueness is proved in [8]. Assume that continuous nondecreasing functions $\varphi_n : \mathbb{R} \to \mathbb{R}$ with $\varphi_n(0) = 0$ are given for $n = 1,2,\ldots,\infty = \mathbb{Z}^+ \cup \{\infty\}$ together with initial data $u_{0n} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and let $u_n \in C([0,\infty):L^1(\mathbb{R}^N))$ be the associated unique solution of

$$(1_n) \quad \begin{cases} u_{nt} - \Delta\varphi_n(u_n) = 0 & \text{in} \quad \mathcal{D}'((0,\infty) \times \mathbb{R}^N) \\ u_n(0,x) = u_{0n}(x), & x \in \mathbb{R}^N. \end{cases}$$

If $\varphi_n \to \varphi_\infty$ and $u_{0n} \to u_{0\infty}$ in suitable ways, we will prove that $u_n \to u_\infty$. However, the precise situation is rather complicated. If $N \geq 3$, there are (at least) two distinct possibilities depending on the behavior of $\varphi_\infty(r)$ near $r = 0$. Roughly speaking, if the graph of $\varphi_\infty(r)$ is too steep near $r = 0$, the associated diffusion is so strong that the convergence of u_n to u_∞ near $|x| = \infty$ is not as good as it is in the case where $\varphi_\infty(r)$ approaches 0 more rapidly as $r \to 0$. To describe the condition on the behavior

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of φ_∞ near zero, let $\beta_\infty = \varphi_\infty^{-1}$. Then β_∞ is a monotone graph which is a function only if φ_∞ is strictly increasing. Let $\beta_\infty^0(r)$ be the element of least modulus of $\beta_\infty(r)$ if $r \in \mathbb{R}(\varphi_\infty)$ (the range of φ_∞). E.g., if $r > 0$ and $r = \varphi_\infty(s)$ for some s, then $\beta_\infty^0(r) = \min\{\tau : r = \varphi_\infty(\tau)\}$. If $r > \varphi_\infty(s)$ $(r < \varphi_\infty(s))$ for all s we set $\beta_\infty^0(r) = \infty$ (respectively, $\beta_\infty^0(r) = -\infty$). With these conventions we have:

<u>Theorem</u>: Let $u_n \in C([0,\infty]:L^1(\mathbb{R}^N)) \cap L^\infty((0,\infty) \times \mathbb{R}^N)$ be the solution of (1_n), $n = 1,2,\ldots,\infty$,

(2) $$\lim_{n\to\infty} \varphi_n(r) = \varphi_\infty(r) \quad \text{for} \quad r \in \mathbb{R},$$

and

(3) $$\lim_{n\to\infty} \|u_{0n} - u_{0\infty}\|_{L^1(\mathbb{R}^N)} = 0 .$$

Then the following assertions hold:

(i) If $N = 1$ or $N = 2$, then $u_n \to u_\infty$ in $C([0,\infty):L^1(\mathbb{R}^N))$.

(ii) If $N \geq 3$ and

(4) $$-\int_a^\infty r^{N-1}\beta_\infty^0\left(\frac{-1}{r^{N-2}}\right)dr = \int_a^\infty r^{N-1}\beta_\infty^0\left(\frac{1}{r^{N-2}}\right)dr = \infty \quad \text{for } a > 0,$$

then $u_n \to u_\infty$ in $C([0,\infty):L^1(\mathbb{R}^N))$.

(iii) If $N \geq 3$, $0 < \alpha$ and $\rho_\alpha(x) = (1 + |x|^2)^{-\alpha}$ then $\rho_\alpha u_n \to \rho_\alpha u_\infty$ in $C([0,\infty):L^1(\mathbb{R}^N))$.

To clarify the nature of the condition (4), let $\varphi_\infty(r) = |r|^m \operatorname{sign} r$ where $m > 0$. Then β_∞ is a function and $\beta_\infty(r) = |r|^{1/m}\operatorname{sign} r$. The condition (4) is fulfilled exactly when $(N-1) - (N-2)/m \geq -1$ or $m \geq (N-2)/N$. It will be shown in Section 3 that if $0 < m < (N-2)/N$ the conclusion of (ii) fails in general. However, even if $0 < m < (N-2)/N$ we still have (iii) holding. The behavior near $|x| = \infty$ is adquately damped by the weights ρ_α.

Parts (i) and (ii) of the above theorem are proved in Section 1. The proof of (iii), formulated in an appropriately general way, is given in Section 2. In fact, we associate an m-accretive operator in the corresponding weighted L^1 space with each problem (1) and show this operator depends continuously on φ. Section 3 establishes the necessity of (4) in the class of nonlinearities $\varphi_\infty(r) = |r|^m\operatorname{sign} r$, $0 < m$.

There appears to be little previous work on continuity of solutions of (1) in φ.

The problem (1) with $\varphi(u) = |u|^m \operatorname{sign} u$ and $u_0 \geq 0$ is discussed in, e.g., [13], [15] via replacing u_0 by $u_0 + \varepsilon > 0$ and letting ε tend to 0. The point of this method is to deal with strictly positive approximations for which the problem is nonsingular (since φ' is bounded above and away from zero on the approximations.) In our framework, this corresponds to introducing the approximation $\varphi_\varepsilon(r) = \varphi(r+\varepsilon) - \varphi(\varepsilon)$ of φ. A simpler (in some sense) approximation is $\varphi_\varepsilon(r) = \varphi(r) + \varepsilon r$, which yields nonsingular problems if $m \geq 1$. Moreover, it permits u_0 to change sign. The question of the dependence on the nonlinearity of solutions of problems related to ours appears in the papers [3], [11], [12], [14], but the results of these works and the current paper have nothing in common. The requirement (4) plays a strong role in [4] in the study of the asymptotic behavior of solutions of an associated time-independent problem.

1. Continuity in $C([0,\infty):L^1(\mathbb{R}^N))$.

We begin by a review of the material we will draw upon in the sequel. In order to discuss $u_t - \Delta\varphi(u) = 0$ within the nonlinear semigroup theory we first associate an m-accretive operator A_φ in $L^1(\mathbb{R}^N)$ with the formal expression $A_\varphi u = -\Delta\varphi(u)$. This is done via the results of [5]. Let φ be a maximal monotone graph in $\mathbb{R}$ (see [7]) and consider the problem

$$(1.1) \qquad u - \Delta w = f \text{ in } D'(\mathbb{R}^N),\ w(x) \in \varphi(u(x)) \text{ a.e. } x \in \mathbb{R}^N$$

where $f \in L^1(\mathbb{R}^N)$. The following theorem holds ([5]):

<u>Theorem 1</u> Let φ be a maximal monotone graph in $\mathbb{R}$, $0 \in \varphi(0)$ and $f \in L^1(\mathbb{R}^N)$. Then:

(i) If $N \geq 3$ there exists unique $u \in L^1(\mathbb{R}^N)$ and $w \in M^{N/(N-2)}(\mathbb{R}^N)$ for which (1.1) holds.

(ii) If $N = 2$ and $0 \in \operatorname{int} D(\varphi)$, there is a unique $u \in L^1(\mathbb{R}^2)$ for which there exists $w \in L^1_{loc}(\mathbb{R}^2)$ with $w(x) \in \varphi(u(x))$ a.e., $\operatorname{grad} w \in M^2(\mathbb{R}^2)^2$ and (1.1) is satisfied.

(iii) If $N = 1$ and $0 \in \operatorname{int} D(\varphi)$, there is a unique $u \in L^1(\mathbb{R})$ for which there exists $w \in L^1_{loc}(\mathbb{R})$ with $w(x) \in \varphi(u(x))$ a.e. and (1.1) is satisfied.

<u>Remarks:</u> See [5, appendix] concerning the spaces $M^p(\mathbb{R}^N)$. To relate (1.1) to the problem studied in [5], put $\beta = \varphi^{-1}$ and rewrite (1.1) as $\beta(w) - \Delta w \ni f$.

The operators A_φ may now be defined by setting $A_\varphi u = \{g \in L^1(\mathbb{R}^N): u$ is the solution of (1.1) for $f = g + u\}$ for each $u \in L^1(\mathbb{R}^N)$. The results of [5] also contain:

<u>Proposition 2</u> Under the assumptions of Theorem 1, (i), (ii), (iii), A_φ is m-accretive in $L^1(\mathbb{R}^N)$. Moreover $(I + A_\varphi)^{-1} = J_\varphi : L^1(\mathbb{R}^N) \to L^1(\mathbb{R}^N)$ has the following properties:

(i) J_φ is a translation and rotation invariant contraction on $L^1(\mathbb{R}^N)$.

(ii) $\forall f,g \in L^1(\mathbb{R}^N)$, $f \leq g$ a.e. implies $J_\varphi f \leq J_\varphi g$ a.e.

(iii) $\forall f \in L^1(\mathbb{R}^N)$, $-\|f^-\|_{L^\infty(\mathbb{R}^N)} \leq J_\varphi f \leq \|f^+\|_{L^\infty(\mathbb{R}^N)}$ a.e. where $f^+ = \max(f,0)$, $f^- = -\min(f,0)$.

Moreover, if $f, u = J_\varphi f, w$ are related as in Theorem 1, we have:

(iv) If $N \geq 3$ there is a constant C_N depending only on N such that

$$\|w\|_{M^{N/(N-2)}(\mathbb{R}^N)} \leq C_N \|f\|_{L^1(\mathbb{R}^N)}.$$

(v) If $N \geq 2$ there is a constant D_N depending only on N such that

$$\| |\text{grad}\, w| \|_{M^{N/(N-2)}(\mathbb{R}^N)} \leq D_N \|f\|_{L^1(\mathbb{R}^N)} .$$

In view of these results, the problem $u_t - \Delta\varphi(u) = 0$, $u(0,x) = u_0(x)$ may be transcribed as

(1.2) $$\frac{du}{dt} + A_\varphi u = 0 , \; u(0) = u_0$$

and solved by the nonlinear semigroup theory. (See, e.g., [2], [9],[10]). If φ is continuous and $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ the resulting solution satisfies $u \in C([0,\infty):L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty) \times \mathbb{R}^N)$ and (1) of the introduction. (While this is not in print, it is easy to show.) In the general case, we are interested in the continuity of the solution of (1.2) with respect to φ. For this we will employ the result of nonlinear semigroup theory which states that if each of A_n, $n = 1,2,\ldots,\infty$ is an m-accretive operator in a Banach space X, $x_n \in \overline{D(A_n)}$ and u_n is the solution of

(1.4) $$\frac{du_n}{dt} + A_n u_n = 0 , \; u_n(0) = x_n$$

then $A_n \to A_\infty$, $x_n \to x_\infty$ implies $u_n \to u_\infty$ in $C([0,\infty):X)$ where $A_n \to A$ means

(1.5) $$\lim_{n\to\infty} (I+A_n)^{-1}x = (I+A_\infty)^{-1}x \quad \text{for } x \in X .$$

(See, e.g., [9], [10] for statements and references.) Thus we are reduced to studying the question of when $\varphi_n \to \varphi_\infty$ (as m-accretive operators in $\mathbb{R}$) implies $A_{\varphi_n} \to A_{\varphi_\infty}$ as m-accretive operators in $L^1(\mathbb{R}^N)$.

Remark: The convergence $\varphi_n \to \varphi_\infty$ can be rephrased as $\varphi_n^0(r) \to \varphi_\infty^0(r)$ a.e. $r \in \mathbb{R}$, where $\varphi_j^0(r)$ denotes the element of $\varphi_j(r)$ of least modulus and $\varphi_j^0(r) = \infty(-\infty)$ if r is above (respectively, below) $D(\varphi_j)$.

The main result of this section is:

Theorem 3. Let φ_n, $n = 1,2,\ldots,\infty$ be maximal monotone graphs in $\mathbb{R}$ with $0 \in \varphi_n(0)$. Let $\varphi_n \to \varphi_\infty$ as $n \to \infty$. If

(1.6) $\quad N = 1$ or $N = 2$ and $0 \in \text{int}\, D(\varphi_\infty)$,

or

$$(1.7)\qquad N \geq 3\ ,\quad \beta_\infty = \varphi_\infty^{-1} \text{ and}$$

$$-\int_a^\infty r^{N-1}\ \beta_\infty^o\left(\frac{-1}{r^{N-2}}\right)dr = \int_a^\infty r^{N-1}\ \beta_\infty^o\left(\frac{1}{r^{N-2}}\right)dr = \infty$$

for $a > 0$

then $A_{\varphi_N} \to A_{\varphi_\infty}$ as m-accretive operators in $L^1(\mathbb{R}^N)$.

If $\varphi_\infty(r) = |r|^m \operatorname{sign} r$, we saw that (1.7) is equivalent to $(N-2)/N \leq m$. Another example which helps fix the ideas is $\varphi_\infty(r) \equiv 0$. Then $\beta_\infty^o(r) = \infty$ for $r > 0$, $-\infty$ for $r < 0$ and 0 for $r = 0$ and (1.7) holds. Also, if $\varphi_\infty(r) = \mathbb{R}$ for $r = 0$ and $\varphi_\infty(r) = \emptyset$ for $r \neq 0$, then $\beta_\infty(r) \equiv 0$. In this case (1.7) fails and so does the conclusion. If $\varphi_n(r) = nr$, then $\varphi_n \to \varphi_\infty$. However, the solution of $u_n + A_{\varphi_n} u_n = u_n - n\Delta u_n = f$ satisfies $\int_{\mathbb{R}^N} u_n = \int_{\mathbb{R}^N} f$ for $n = 1,2,\ldots$ while the solution of $u_\infty + A_{\varphi_\infty} u_\infty \ni f$ is $u_\infty \equiv 0$. Thus $u_n \to u_\infty$ in $L^1(\mathbb{R}^N)$ is impossible if $\int_{\mathbb{R}^N} f \neq 0$.

<u>Proof of Theorem 3 for $N \geq 3$</u>. The principle new step in the proof is:

<u>Proposition 4</u>. Let φ_n, $n = 1,2,\ldots,\infty$ be as in Theorem 3. Let (1.7) hold and $f \in L^1(\mathbb{R}^N)$. Then $\{(I + A_{\varphi_n})^{-1} f : n = 1,2,\ldots,\infty\}$ is precompact in $L^1(\mathbb{R}^N)$.

<u>Proof</u>: First let φ be an arbitrary maximal monotone graph in $\mathbb{R}$ with $0 \in \varphi(0)$ and $J_\varphi = (I + A_\varphi)^{-1}$. Let $\tau_y g(x) = g(x+y)$ for $y \in \mathbb{R}^N$. Since we have that $J_\varphi 0 = 0$, $\tau_y J_\varphi = J_\varphi \tau_y$ and J_φ is a contraction on $L^1(\mathbb{R}^N)$ (Proposition 2) we also have

$$(1.8)\qquad \begin{cases} \|J_\varphi f\|_{L^1(\mathbb{R}^N)} \leq \|f\|_{L^1(\mathbb{R}^N)} \\ \\ \|\tau_y J_\varphi f - J_\varphi f\|_{L^1(\mathbb{R}^N)} = \|J_\varphi \tau_y f - J_\varphi f\|_{L^1(\mathbb{R}^N)} \leq \|\tau_y f - f\|_{L^1(\mathbb{R}^N)} \end{cases}$$

which shows that $\{J_\varphi f : \varphi \text{ a maximal monotone graph in } \mathbb{R} \text{ with } 0 \in \varphi(0)\}$ is a bounded subset of $L^1(\mathbb{R}^N)$ on which translations are equicontinuous. This implies precompactness in $L^1_{loc}(\mathbb{R}^N)$. We need then only show

$$(1.9)\qquad \lim_{R\to\infty} \int_{\{|x|\geq R\}} |J_{\varphi_n} f(x)|\,dx = 0 \text{ uniformly for } n = 1,2,\ldots,\infty.$$

Let $\varepsilon > 0$ and choose $M, R_0 > 0$ such that

(1.10) $$\|(f - M\chi_{\{|x| \geq R_0\}})^+\|_{L^1(\mathbb{R}^N)} < \varepsilon$$

where $r^+ = \max(r,0)$ and χ_A is the characteristic function of $A \subseteq \mathbb{R}^N$. Set $g = M\chi_{\{|x| \geq R_0\}}$. We have $f \leq f^+ \leq g + (f-g)^+$ and so

$$(J_{\varphi_n} f)^+ \leq J_{\varphi_n} f^+ \leq J_{\varphi_n}(g + (f-g)^+)$$

because J_{φ_n} is order preserving (Proposition 2). Using this and (1.10),

$$\int_{\{|x| \geq R\}} (J_{\varphi_n} f)^+ \leq \int_{\{|x| \geq R\}} J_{\varphi_n}(f^+) \leq \int_{\{|x| \geq R\}} J_{\varphi_n}(g + (f-g)^+)$$

$$\leq \int_{\{|x| \geq R\}} J_{\varphi_n} g + \int_{\{|x| \geq R\}} |J_{\varphi_n}(g + (f-g)^+) - J_{\varphi_n}(g)|$$

$$\leq \int_{\{|x| \geq R\}} J_{\varphi_n} g + \|(f-g)^+\|_{L^1(\mathbb{R}^N)} \leq \int_{\{|x| \geq R\}} J_{\varphi_n} g + \varepsilon,$$

since J_{φ_n} is an $L^1(\mathbb{R}^N)$ contraction. Recalling that $\varepsilon > 0$ is arbitrary, we conclude that it is enough to show

(1.12) $$\lim_{R\to\infty} \int_{\{|x| \geq R\}} J_{\varphi_n} g = 0 \text{ uniformly for } n = 1,2,\ldots,\infty$$

in order to establish the same for $(J_{\varphi_n} f)^+$. Treating $(J_{\varphi_n} f)^-$ in a similar way, (1.10) holds for f if it holds for $g = \pm M\chi_{\{|x| \leq R_0\}}$, $M > 0$, $R_0 > 0$. Thus, without loss of generality, we assume

(1.13) $$f(x) = \begin{cases} M & \text{if } |x| \leq R_0 \\ 0 & \text{if } |x| > R_0 \end{cases}$$

where $M > 0$. The case $M < 0$ is entirely similar. We proceed now by giving some estimates on $J_\varphi f$ for arbitrary φ and f given by (1.13). Since J_φ is rotation invariant there are nonnegative functions $u(r)$, $w(r)$ for $r > 0$ such that

(1.14) $$\begin{cases} r^{N-1}u(r) - (r^{N-1}w'(r))' = r^{N-1}M\chi_{\{0 < r < R_0\}} \\ w(r) \in \varphi(u(r)) \text{ a.e.} \end{cases}$$

and $J_\varphi f(x) = u(|x|)$. (Observe that we are abusing notation a bit by using $u(r)$, $w(r)$ to denote the functions of a single variable corresponding to the rotation invariant functions u,w of N variables in Theorem 1 and Proposition 2.) The next task is to estimate $\int_R^\infty r^{N-1}u(r)dr$ (which differs from $\int_{\{|x|\geq R\}} |J_\varphi f|$ by a constant factor). Letting $\beta = \varphi^{-1}$ we rewrite (1.14) as

$$r^{N-1}\beta(w(r)) - (r^{N-1}w'(r))' = r^{N-1}M\chi_{\{0 < r < R_0\}} \tag{1.15}$$

and hereafter usually ignore the possibility that β is multivalued for notational simplicity. Let

$$H = \frac{1}{\omega_N}\int_{\mathbb{R}^N} f = \frac{1}{N} R_0^N M \tag{1.16}$$

where ω_N is the area of the unit sphere in $\mathbb{R}^N$. The relation $\int_{\mathbb{R}^N} J_\varphi f \leq \int_{\mathbb{R}^N} f$ is equivalent to

$$\int_0^\infty r^{N-1} \beta(w(r))dr \leq H . \tag{1.17}$$

In addition, the estimate of Proposition 2 (iv) (see [5, Appendix]) implies

$$\int_r^{2r} (w(s))s^{N-1}ds \leq c_N H(\int_r^{2r} s^{N-1}ds)^{2/N} \leq c_N Hr^2 \tag{1.18}$$

where c_N will denote various constants depending only on N. Since (1.15) implies

$$(r^{N-1}w'(r))' = r^{N-1}\beta(w(r)) \geq 0 \quad \text{for} \quad r > R_0 , \tag{1.19}$$

$r^{N-1}w'(r)$ is nondecreasing on (R_0,∞). Thus if $r_0^{N-1}w'(r_0) > 0$ for some $r_0 \in (R_0,\infty)$, w' will be strictly positive on (r_0,∞) and $w(+\infty) > 0$. However, (1.18) then implies that if $r > r_0$

$$w(r) \int_r^{2r} s^{N-1}ds \leq \int_r^{2r} w(s)s^{N-1}ds \leq c_n H\, r^2$$

and $w(r) \leq c_N\, H\, r^{2-N}$ and so $w(+\infty) = 0$, a contradiction. Thus we have

$$r^{N-1}w'(r) \leq 0 \quad \text{for} \quad r \geq R_0 . \tag{1.20}$$

Next we integrate the inequality $t^{N-1}w'(t)/s^{N-1} \geq w'(s)$ (which is valid for $R_0 \leq s \leq t$) over $r \leq s \leq t$ to find

$$w(r) \geq w(t) + \frac{1}{N-2}\left(\frac{1}{r^{N-2}} - \frac{1}{t^{N-2}} \right)(-t^{N-1}w'(t)), \; R_0 \leq r \leq t \tag{1.21}$$

and so, choosing $t = 2r$ and using $w \geq 0$,

$$(1.22)\qquad w(r) \geq c_N \frac{1}{r^{N-2}}(-(2r)^{N-1}w'(2r)), \qquad R_0 \leq r .$$

Now we set $\varphi = \varphi_n$ and correspondingly write β_n, w_n, etc. Let $\varepsilon > 0$ be given. It follows from (1.7) that there is an $R(\varepsilon) > R_0$ such that

$$(1.23)\qquad \int_{R_0}^{R(\varepsilon)} r^{N-1}\beta_\infty^o\left(\frac{\varepsilon}{r^{N-2}}\right)dr > 2H .$$

The convergence $\varphi_n \to \varphi_\infty$ implies $\beta_n = \varphi_n^{-1} \to \beta_\infty$. By Fatou's lemma we obtain from (1.23) and $\beta_n \to \beta_\infty$ the existence of an integer $M(\varepsilon)$ such that

$$(1.24)\qquad \int_{R_0}^{R(\varepsilon)} r^{N-1}\beta_n^o\left(\frac{\varepsilon}{r^{N-2}}\right)dr > 2H \quad \text{for} \quad n \geq M(\varepsilon) .$$

Using (1.17), (1.20), (1.22), that $-r^{N-1}w'(r)$ is nonincreasing and the monotonicity of β_n we have

$$\int_{R_0}^{R(\varepsilon)} r^{N-1}\beta_n\left(\frac{c_N(-(2R(\varepsilon))^{N-1}w_n'(2R(\varepsilon))}{r^{N-2}}\right)dr \leq H .$$

But then by (1.24) and the monotonicity of β_n,

$$(1.25)\qquad 0 \leq -R^{N-1}w_n'(R) \leq -R(\varepsilon)^{N-1}w_n'(R(\varepsilon)) \leq c_N\varepsilon \quad \text{for} \quad R \geq R(\varepsilon),\ n \geq M(\varepsilon) .$$

We are now prepared to demonstrate (1.12), which is equivalent to

$$(1.26)\qquad \lim_{R\to\infty}\int_R^\infty r^{N-1}\beta_n(w(r))dr = 0 \quad \text{uniformly for} \quad n = 1,2,\ldots .$$

Indeed, by (1.19), if $k_n(r) = r^{N-1}w_n'(r)$ we have

$$\int_R^\infty r^{N-1}\beta(w_n(r))dr = k_n(\infty) - k_n(R) .$$

However, by (1.25), $k_n(\infty) - k_n(R) \leq c_N\varepsilon$ if $n \geq M(\varepsilon)$ and $R \geq R(\varepsilon)$. Thus, if $\delta > 0$, we can guarantee

$$(1.27)\qquad \int_R^\infty r^{N-1}\beta(w_n(r))dr \leq \delta$$

provided only that n and R are large enough. For the remaining finite number of indices n, (1.27) also holds if only R is sufficiently large and the proof is complete.

<u>End of Proof of Theorem 3 for $N \geq 3$:</u> Fix $f \in L^1(\mathbb{R}^N)$ and let φ_n, $n = 1,2,\ldots,\infty$ satisfy the conditions of Theorem 3. By Proposition 3, $\{J_{\varphi_n}f\}$ is precompact in $L^1(\mathbb{R}^N)$. It is enough to show that if n_k, $k = 1,2,\ldots$ is a subsequence of $1,2,\ldots$ then $J_{\varphi_{n_k}}f \to u$ in $L^1(\mathbb{R}^N)$ implies $u = J_{\varphi_\infty}f$. Thus we assume $u_n = J_{\varphi_n}f \to u$ in $L^1(\mathbb{R}^N)$. Let $w_n \in M^{N/(N-2)}(\mathbb{R}^N)$, $w_n(x) \in \varphi_n(u_n(x))$ a.e. and $u_n - \Delta w_n = f$ (as in Theorem 1).

The estimates (iv) and (v) of Proposition 2 imply $\{w_n\}$ is bounded in $W^{1,p}_{loc}(\mathbb{R}^N)$ for $1 \leq p < N/(N-1)$ (see [5, appendix]) and hence we can assume $w_n \to w$ in $L^1_{loc}(\mathbb{R}^N)$. But then $w(x) \in \varphi_\infty(u(x))$ a.e., $w \in M^{N/(N-2)}(\mathbb{R}^N)$ (by Proposition 4 (iv))) and $u - \Delta w = f$ in $\mathcal{D}'(\mathbb{R}^N)$. Thus $u = J_\varphi f$ by Theorem 1. The proof is complete.

Proof of Theorem 3 for $N = 1,2$. The proof is analogous to the case $N \geq 3$ in many respects and we only sketch it here. The analogue of Proposition 4 is:

Proposition 5: Let $\delta > 0$, $r_0 > 0$ and $\Phi = \{$maximal monotone graphs in $\mathbb{R}$ with $0 \in \varphi(0)$ and $|\varphi^o(\pm\delta)| < r_0\}$. Let $N \in \{1,2\}$ and $f \in L^1(\mathbb{R}^N)$. Then

$$\{(I + A_\varphi)^{-1}f: \varphi \in \Phi\}$$

is precompact in $L^1(\mathbb{R}^N)$.

Proof. As before we reduce to the case $f = M\chi_{\{|x|\leq R_0\}}$ and the problem of estimating

$$\int_R^\infty r^{N-1}\beta(w(r))dr = \int_R^\infty r^{N-1} u(r)dr$$

where $J_\varphi f(x) = u(|x|)$, $w(r) \in \varphi(u(r))$ or $u(r) \in \beta(w(r))$ and

$$r^{N-1}\beta(w(r)) - (r^{N-1}w'(r))' = M\chi_{\{0 \leq r < R_0\}} \quad (1.23)$$

The condition on φ corresponds to

$$\delta \leq |\beta(\pm r_0)| \quad (1.24)$$

Integrating $w'(r) \geq (s/r)^{N-1}w'(s)$ with respect to r yields

$$\left\{\begin{array}{ll} w(r) \geq w(s) + (r-s)w'(s), & R_0 \leq s \leq r,\ N = 1 \\ w(r) \geq w(s) + s\ln(r/s)w'(s), & R_0 \leq s \leq r,\ N = 2\ . \end{array}\right. \quad (1.25)$$

Thus either $w'(s) \leq 0$ or $w(\infty) = \infty$. But $w(\infty) = \infty$ is inconsistant with (1.24) and

$$\int_0^\infty r^{N-1}\ \beta(w(r)dr \leq H\ . \quad (1.26)$$

Thus $w' \leq 0$ on (R_0,∞). Now we integrate $(r/s)^{N-1}w'(r) \geq w'(s)$ with respect to s to find the analogues of (1.20):

$$\left\{\begin{array}{ll} w(r) \geq w(t) + (t-r)(-w'(t)), & R_0 \leq r \leq t,\ N = 1 \\ w(r) \geq w(t) + \ln(t/r)[-tw'(t)], & R_0 \leq r \leq t,\ N = 2\ . \end{array}\right. \quad (1.27)$$

Now the monotonicity of $w(r)$ and hence $\beta(w(r))$ implies

$$\beta(w(2r))\int_r^{2r} s^{N-1}ds \leq \int_0^\infty s^{N-1}\ \beta(w(s))ds \leq H \quad \text{for } r \geq 2R_0$$

so we have

(1.28) $\beta(w(r)) \le c_N r^{-N} H$ for $r \ge 2R_0$.

If r is large enough to guarantee $c_N r^{-N} H < \delta$, we find $w(r) \le r_0$ from (1.28) and (1.24). Thus (1.27) implies $-(t^{N-1}w'(t)) \to 0$ as $t \to \infty$ uniformly for $\varphi \in \Phi$ and the proof is completed as before.

<u>End of proof of Theorem 3 for N = 1,2.</u> The convergence $\varphi_n \to \varphi_\infty$ and $0 \in \operatorname{int} D(\varphi_\infty)$ implies that there are $\delta, r_0 > 0$ for which $\varphi_n \in \Phi$ as defined in Proposition 5 if only n is sufficiently large. Hence for $f \in L^1(\mathbb{R}^N)$ $\{J_{\varphi_n} f : n = 1,2,\ldots\}$ is precompact. The proof is completed as in the case $N \ge 3$. However we must use that $\{w_n = J_{\varphi_n} f : n = 1,2,\ldots\}$ is bounded in $L^1_{loc}(\mathbb{R}^N)$ in addition to the information in Proposition 2. See the proofs of [5, Thms 3.1 & 4.1].

We will record one further result here for future reference:

<u>Proposition 6.</u> Let φ be a maximal monotone graph in $\mathbb{R}$ with $0 \in \varphi(0)$. If $N = 1$ or $N = 2$ assume, in addition, that $0 \in \operatorname{int} D(\varphi)$. Let A_φ be the associated m-accretive operator in $L^1(\mathbb{R}^N)$. Then

$$\overline{D(A_\varphi)} = \{v \in L^1(\mathbb{R}^N) : v(x) \in \overline{D(\varphi)} \text{ a.e. }\}$$

where $\overline{D(A_\varphi)}$ is the closure of $D(A_\varphi)$ in $L^1(\mathbb{R}^N)$.

<u>Proof:</u> We must show that if

(1.29) $v \in L^1(\mathbb{R}^N)$ and $\inf D(\varphi) \le v(x) \le \sup D(\varphi)$ a.e. $x \in \mathbb{R}^N$

then $v \in \overline{D(A_\varphi)}$. Let $\lambda > 0$ and set $u_\lambda = (I + \lambda A_\varphi)^{-1} v = (I + A_{\lambda\varphi})^{-1} v = J_{\lambda\varphi} v$. Now $u_\lambda \in D(A_\varphi)$, and we will establish that $u_\lambda \to v$ $L^1(\mathbb{R}^N)$ as $\lambda \downarrow 0$, whence the result. One has $\lambda\varphi \to \varphi_0$ as $\lambda \downarrow 0$ where

$$(1.30) \qquad \varphi_0(r) = \begin{cases} \{0\} & \inf D(\varphi) < r < \sup D(\varphi) \\ [-\infty, 0] & r = \inf D(\varphi) \\ [0, \infty) & r = \sup D(\varphi) \\ \phi & \text{otherwise,} \end{cases}$$

is the "subdifferential of the indicator function" of $\overline{D(\varphi)}$. Since $(I + A_{\varphi_0})^{-1} v = v$ by (1.29), (1.30), Theorem 3 implies $u_\lambda \to v$ *if* φ_0 satisfies the hypotheses on φ_∞ in Theorem 3. This is clearly the case if $\inf D(\varphi) < 0 < \sup D(\varphi)$, while the case $\inf D(\varphi) = \sup D(\varphi) = 0$ (which is allowed for $N \ge 3$) is trivial. If, for example, $\inf D(\varphi) < \sup D(\varphi) = \max D(\varphi) = 0$, (which is possible for $N \ge 3$), let $\tilde{\varphi}(r) = \varphi(r)$ for $r < 0$,

$\tilde{\varphi}(0) = \varphi(0) \cap (-\infty, 0]$, $\tilde{\varphi}(r) = \{0\}$ for $r > 0$. Then, since $J_{\lambda\tilde{\varphi}}v \le J_{\lambda\tilde{\varphi}}0 = 0$ by (1.29) and Proposition 2, and $\tilde{\varphi}(r) \subset \varphi(r)$ for $r \le 0$, $J_{\lambda\tilde{\varphi}}v = J_{\lambda\varphi}v$. The above argument applied to $\tilde{\varphi}$ in place of φ yields $J_{\lambda\varphi}v = J_{\lambda\tilde{\varphi}}v \to v$ in $L^1(\mathbb{R}^N)$ as $\lambda \downarrow 0$. The remaining case $\inf D(\varphi) = \min D(\varphi) = 0$ is exactly the same.

2. Continuity in $L^1(\rho_\alpha)$ for $N \geq 3$.

Throughout this section we assume $N \geq 3$. The weights ρ_α are given by

(2.1) $$\rho_\alpha(x) = (1 + |x|^2)^{-\alpha}.$$

$L^1(\rho_\alpha)$ denotes the weighted L^1-space determined by the norm

(2.2) $$\|u\|_{L^1(\rho_\alpha)} = \int_{\mathbb{R}^N} \rho_\alpha(x)|u(x)|dx.$$

The main result of this section is:

<u>Theorem 7.</u> Let φ be a maximal monotone graph in $\mathbb{R}$ with $0 \in \varphi(0)$. Let $0 < \alpha \leq (N-2)/2$. Then the operator A_φ in $L^1(\rho_\alpha)$ defined by

(2.3) $$A_\varphi u = \{-\Delta w : w \in L^1(\rho_{\alpha+1}),\ -\Delta w \in L^1(\rho_\alpha) \text{ and } w(x) \in \varphi(u(x)) \text{ a.e.}\}$$

for $u \in L^1(\rho_\alpha)$ is m-accretive in $L^1(\rho_\alpha)$. Moreover, if φ_n, $n = 1,2,\ldots,\infty$ are maximal monotone graphs in $\mathbb{R}$ with $0 \in \varphi_n(0)$ and $\varphi_n \to \varphi_\infty$ (as maximal monotone graphs) as $n \to \infty$, then $A_{\varphi_n} \to A_{\varphi_\infty}$ as m-accretive operators in $L^1(\rho_\alpha)$.

As a preliminary to the proof of Theorem 7 we consider the problem $-\Delta w = f \in L^1(\rho_\alpha)$. Let

(2.3) $$E_N(x) = a_N/|x|^{N-2}$$

where a_N is that constant for which $-\Delta E_N = \delta_0$, the Dirac mass at the origin. That is, E_N is a fundamental solution of $-\Delta$.

<u>Proposition 8.</u> Let $0 < \alpha \leq \frac{N-2}{2}$ and $f \in L^1(\rho_\alpha)$. Then there is exactly one solution $w \in L^1(\rho_{\alpha+1})$ of $-\Delta w = f$. This solution is given by

(2.4) $$w(x) = E_N * f(x) = \int_{\mathbb{R}^N} \frac{a_N}{|x-y|^{N-2}} f(y)dy.$$

Moreover, grad $w \in L^1(\rho_{\alpha+1/2})^N$ and

(2.5) $$\|E_N * f\|_{L^1(\rho_{\alpha+1})} + \|\,|\text{grad}(E_N * f)|\,\|_{L^1(\rho_{\alpha+\frac{1}{2}})} \leq c_N \|f\|_{L^1(\rho_\alpha)}$$

for a constant c_N depending only on N.

<u>Proof:</u> There are two main points whose proofs we will sketch. First we claim that if $w \in L^1(\rho_{\alpha+1})$ and $-\Delta w = 0$, then $w = 0$. This is observed, for example, in the proof of the remark following Theorem 2.1 in [5]. Alternatively, $L^1(\rho_{\alpha+1})$ is a subspace of $\mathcal{S}'(\mathbb{R}^N)$, the continuous linear functionals on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decreasing C^∞ functions. Hence $w \in L^1(\rho_{\alpha+1})$ and $\Delta w = 0$ implies w is a constant.

But if $\alpha \leq (N-2)/2$, the only constant function in $L^1(\rho_{\alpha+1})$ is 0. This establishes the uniqueness claim of the proposition. The existence, (2.4) and the estimate (2.5) all follow easily if we establish (2.5). Observe that

$$\|E_N * f\|_{L^1(\rho_{\alpha+1})} \leq \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{a_N}{|x-y|^{N-2}} (1+|x|^2)^{-(\alpha+1)} dx |f(y)| dy \right) \tag{2.6}$$

and

$$\|(\text{grad}\, E_N) * f\|_{L^1(\rho_{\alpha+1/2})} \leq \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} \frac{(N-2)a_N}{|x-y|^{N-1}} (1+|x|^2)^{-(\alpha+\frac{1}{2})} dx |f(y)| dy \right) \tag{2.7}$$

To estimate these quantities appropriately we use:

Lemma 9. Let $\beta, \gamma > 0$, $\beta+\gamma-N > 0$ and $N > \beta$.

Then there is a constant $C = C(\beta,\gamma,N)$ such that

$$\int \frac{1}{|x-y|^{\beta}} \frac{1}{1+|x|^{\gamma}} dx \leq \frac{C}{1+|y|^{\delta}}$$

where $\delta = \min(\beta+\gamma-N, \beta)$.

Sketch of Proof of Lemma 9. Using the assumptions that $\beta+\gamma-N>0$ for large $|x|$ and that $N > \beta$ for x near y one sees

$$g(y) = \int \frac{1}{|x-y|^{\beta}} \frac{1}{1+|x|^{\gamma}} dx < \infty$$

and g is bounded on compact sets. Decomposing the integral defining g into the sum of three integrals, one over each of the sets $\Omega_1 = \{x \in \mathbb{R}^N : |x-y| \leq \frac{1}{2}|y|\}$, $\Omega_2 = \{x \in \mathbb{R}^N : |x-y| \geq \frac{1}{2}|y|$ and $|x| \leq 2|y|\}$ and $\Omega_3 = \{x \in \mathbb{R}^N : |x| \geq 2|y|\}$ and using that $|x| \geq \frac{1}{2}|y|$ on Ω_1 while $|x-y| \geq \frac{1}{2}|x|$ on Ω_3, one obtains $g(y) \leq C(|y|^{N-(\gamma+\beta)} + |y|^{-\beta})$ where the term $|y|^{-\beta}$ arises from the integral over Ω_2. The result follows.

End of Proof of Proposition 8. From Lemma 9 we find

$$\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{1}{(1+|x|^2)^{\alpha+1}} dx \leq \frac{\text{cons.}}{1+|y|^{\delta}} \leq \frac{\text{cons.}}{(1+|y|^2)^{\delta/2}}$$

and

$$\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-1}} \frac{1}{(1+|x|^2)^{(\alpha+\frac{1}{2})}} dx \leq \frac{\text{cons.}}{1+|y|^{\delta}} \leq \frac{\text{cons.}}{(1+|y|^2)^{\delta/2}}$$

where $\delta = \min(2\alpha, N-2) = \min(2\alpha, N-1) = \alpha$ for $\alpha \leq (N-2)/2$. Together with (2.6) and (2.7), these estimates imply (2.5).

The next ingredient in the proof of Theorem 7 is:

Proposition 10: Let $w \in L^1(\rho_{\alpha+1})$, $\Delta w \in L^1(\rho_\alpha)$ and $0 < \alpha \le (N-2)/N$. Let $p \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $0 \le p'$, $p' \in L^\infty(\mathbb{R})$ and $p(0) = 0$. Then $p'(w)|\text{grad}\, w|^2 \in (L^1(\rho_\alpha))$ and

$$(2.8) \qquad \int_{\mathbb{R}^N} (\Delta w)\, p(w)\rho_\alpha + \int_{\mathbb{R}^N} p'(w)|\text{grad}\, w|^2\rho_\alpha + \int_{\mathbb{R}^N} j(w)(-\Delta\rho_\alpha) \le 0$$

where $j(r) = \int_0^r p(s)ds$.

Proof: Let $\{f_n\} \subset C_0^\infty(\mathbb{R}^N)$, $f_n \to -\Delta w$ in $L^1(\rho_\alpha)$. Then $w_n = E_N * f_n$ satisfies (by Proposition 8)

$$\begin{aligned} &\text{(i)} \quad w_n \to w \text{ in } L^1(\rho_{\alpha+1}), \\ (2.9) \quad &\text{(ii)} \quad \text{grad}\, w_n \to \text{grad}\, w \text{ in } (L^1(\rho_{\alpha+1/2}))^N, \\ &\text{(iii)} \quad -\Delta w_n = f_n \to -\Delta w \text{ in } L^1(\rho_\alpha). \end{aligned}$$

We may integrate the identity

$$\begin{aligned}(\Delta w_n)p(w_n)\rho_\alpha &= \text{div}(p(w_n)\rho_\alpha\, \text{grad}\, w_n - j(w_n)\,\text{grad}\,\rho_\alpha) \\ &\quad - |\text{grad}\, w_n|^2\, p'(w_n)\rho_\alpha + j(w_n)\Delta\rho_\alpha\end{aligned}$$

over the ball $\{x \in \mathbb{R}^N : |x| \le R\} = B_R$ and let $R \to \infty$ to conclude that (2.8) holds with w replaced by w_n. Indeed, since $f_n \in C_0^\infty(\mathbb{R}^N)$, w_n and $\text{grad}\, w_n$ decay like $|x|^{-N+2}$ and $|x|^{-N+1}$, respectively, as $|x| \to \infty$ while $p \in L^\infty$ and $|j(w_n)| \le \|p\|_{L^\infty}(w_n)$. Thus

$$\begin{aligned}|p(w_n)\rho_\alpha\, \text{grad}\, w_n - j(w_n)\text{grad}\,\rho_\alpha| &\le \text{const.}\ (|x|^{-2\alpha}|x|^{-N+1} + |x|^{-N+2}|x|^{-2\alpha-1}) \\ &\le \text{const.}\ (|x|^{-(N-1)+2\alpha})\end{aligned}$$

as $|x| \to \infty$, and the integral over ∂B_R arising from the divergence term above tends to zero as $R \to \infty$. Now we pass to the limit as $n \to \infty$ in the relation

$$\int_{\mathbb{R}^N} (\Delta w_n)p(w_n)\rho_\alpha + \int_{\mathbb{R}^N} p'(w_n)|\text{grad}\, w_n|^2\rho_\alpha + \int_{\mathbb{R}^N} j(w_n)(-\Delta\rho_\alpha) = 0$$

to establish (2.8). The first and third terms above have the desired limiting values by (2.9) (i), (iii) and $|j(w_n)(-\Delta\rho_\alpha)| \le \text{const.}\ |w_n|\rho_{\alpha+1}$, while the second term is handled by Fatou's lemma and (2.9)(ii).

<u>Sketch of Proof of Theorem 7.</u> It is now a (somewhat lengthy) exercise in the use of the arguments of [5] to verify that A_φ as given by (2.3) is m-accretive in $L^1(\rho_\alpha)$. We leave this to the reader. (The third term, $\int_{\mathbb{R}^N} j(w)(-\Delta\rho_\alpha)$, in (2.8) is nonnegative since $-\Delta\rho_\alpha \geq 0$ by direct calculation. It may be dropped while doing the exercise.)

It remains to show that $\varphi_n \to \varphi_\infty$ implies $A_{\varphi_n} \to A_{\varphi_\infty}$ in $L^1(\rho_\alpha)$. Each A_φ has the properties used in the proof of Theorem 3 for $N \geq 3$. In particular, $J_\varphi = (I + A_\varphi)^{-1}$ is order preserving, translation invariant and an $L^1(\rho_\alpha)$-contraction. Thus for each $f \in L^1(\rho_\alpha)$, $\{J_\varphi f: \varphi$ a maximal monotone graph in $\mathbb{R}$ with $0 \in \varphi(0)\}$ is precompact in $L^1_{loc}(\mathbb{R}^N)$. To see this set is precompact in $L^1(\rho_\alpha)$ we need only show

$$\lim_{R\to\infty} \int_{|x|\geq R} \rho_\alpha |J_\varphi f| = 0 \tag{2.10}$$

uniformly in φ for a dense set of f's. But if $f \in L^1(\mathbb{R}^N)$

$$\int_{|x|\geq R} \rho_\alpha \, |J_\varphi(f)| \leq \frac{1}{(1+R^2)^\alpha} \int_{\mathbb{R}^N} |J_\varphi f| \leq \frac{1}{(1+R^2)^\alpha} \|f\|_{L^1(\mathbb{R}^N)}$$

and so (2.10) holds uniformly in φ. The convergence $A_{\varphi_n} \to A_{\varphi_\infty}$ then follows as before.

3. A counterexample for $N \geq 3$.

We consider the special case

$$(3.1) \qquad \begin{cases} u_t - \Delta(u^m) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

of (1) where $\varphi(r) = r^m$, $0 < m < (N-2)/N$. (More precisely, we mean $\varphi(r) = |r|^m \operatorname{sign} r$, but will often write r^m for brevity.) Let $S_m(t)$ be the semigroup on $L^1(\mathbb{R}^N)$ associated with (3.1). With $2^* = 2N/(N-2)$ we will prove:

Proposition 10. Let $0 < m < (N-2)/N$, $\beta = (2-m2^*)/(2^*-2)$ and $u_0 \in L^{\beta+1}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$. Then there is a $T^* > 0$ such that $S_m(t)u_0 = 0$ for $t \geq T^*$.

This has the following relevance for the preceeding results: Set

$$(3.2) \qquad \varphi_n(r) = \begin{cases} \min(nr, r^m) & \text{for } r \geq 0, \\ \max(nr, -|r|^m) & \text{for } r \leq 0 \end{cases}$$

and $n = 1,2,\ldots$. Then $\varphi_n(r) \to |r|^m \operatorname{sign} r$ as maximal monotone graphs as $n \to \infty$. However, one can easily show that if u_n is the solution of

$$(3.3) \qquad \begin{cases} \dfrac{du_n}{dt} + A_{\varphi_n} u_n = 0 \\ u_n(0) = u_0 \end{cases}$$

then $\int_{\mathbb{R}^N} u_n(t) = \int_{\mathbb{R}^N} u_0$ for all $t \geq 0$. It follows from Theorem 3.1 that $u_n(t)$ cannot converge to $S_m(t)u_0$ for any $t \geq T^*$ unless $\int_{\mathbb{R}^N} u_0 = 0$.

Proof of Proposition 10. The formal idea of the proof runs as follows: Multiply the equation $u_t - \Delta u^m = 0$ by $\frac{1}{\beta+1} u^\beta$ and integrate over $\mathbb{R}^N$ to find

$$(3.4) \qquad \frac{d}{dt} \int_{\mathbb{R}^N} u^{\beta+1} + \frac{1}{\beta+1} \int_{\mathbb{R}^N} \nabla u^m \, \nabla u^\beta = 0 .$$

Now the Sobolev inequalities imply

$$(3.5) \qquad \int_{\mathbb{R}^N} \nabla u^m \, \nabla u^\beta = \frac{4m\beta}{(m+\beta)^2} \int_{\mathbb{R}^N} |\nabla u^{\frac{m+\beta}{2}}|^2$$

$$\geq C\left(\int_{\mathbb{R}^N} u^{\frac{m+\beta}{2}2^*}\right)^{2/2^*},$$

where $2^* = 2N/(N-2)$. When $\beta = (2-m2^*)/(2^*-2)$ we have $(m+\beta)2^*/2 = \beta+1$ so, in all, (3.4), (3.5) imply

(3.6) $$\frac{d}{dt}\left(\int_{\mathbb{R}^N} u^{\beta+1}\right) + C\left(\int_{\mathbb{R}^N} u^{\beta+1}\right)^{2/2^*} \le 0 .$$

However, every solution of the inequality

(3.7) $$f' + C f^{\gamma} \le 0 ,$$

where $f:[0,\infty) \to [0,\infty)$, $C > 0$, $0 < \gamma < 1$ has the property that there is a $T^* > 0$ such that $f(t) = 0$ for $t \ge T^*$. This is proved by comparison. The function $g(t) = a(T^*-t)^{\frac{1}{1-\gamma}}$, $a = ((1-\gamma)c)^{\frac{1}{1-\gamma}}$ satisfies

(3.8) $$g' + c\, g^{\gamma} = 0 \quad \text{on } [0,T^*]$$

and $g(0) = aT^{*\frac{1}{1-\gamma}} \ge f(0)$ if T^* is large enough. Hence $f \le g$ on $[0,T^*]$ and $f \equiv 0$ for $t \ge T^*$.

This formal proof can be made rigorous. We sketch how this is done. Assume that $u_0 \in L^1(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$. Let $\lambda > 0$ and define $v_0 = u_0$, and then $v_{i+1} \in L^1(\mathbb{R}^N) \cap L^{\infty}(\mathbb{R}^N)$ by

(3.9) $$\frac{v_{i+1} - v_i}{\lambda} - \Delta v_{i+1}^m = 0$$

or, more precisely, $v_{i+1} = (I + \lambda A_{\varphi})^{-1} v_i$. We multiply (3.9) by v_{i+1}^{β} and use that $v_i\, v_{i+1}^{\beta} \le \frac{1}{\beta+1} v_i^{\beta+1} + \frac{1}{\beta} v_{i+1}^{\beta+1}$ to conclude

$$\frac{1}{\beta+1}\left(v_{i+1}^{\beta+1} - v_i^{\beta+1}\right) - (\Delta v_{i+1}^m) v_{i+1}^{\beta} \le 0 .$$

Integrating this inequality over $\mathbb{R}^N$ yields

$$\frac{1}{\beta+1}\,\frac{1}{\lambda}\left[\int_{\mathbb{R}^N} v_{i+1}^{\beta+1} - \int_{\mathbb{R}^N} v_i^{\beta+1}\right] - \int \nabla v_{i+1}^m\, \nabla v_{i+1}^{\beta} \le 0$$

and so, by Sobolev,

(3.10) $$\frac{1}{\lambda}\,\frac{1}{\beta+1}\left(\int_{\mathbb{R}^N} v_{i+1}^{\beta+1} - \int_{\mathbb{R}^N} v_i^{\beta+1}\right) + C\left(\int (v_{i+1})^{(\frac{m+\beta}{2})2^*}\right)^{2/2^*} \le 0 .$$

The minipulations involving $\int (\Delta v_{i+1}^m) v_{i+1}^{\beta}$ can be justified by [5, Appendix]).
Set $v_{\lambda}(t) = v_i$ for $i\lambda \le t < (i+1)\lambda$. Then the semigroup theory implies $v_{\lambda} \to u$, where u is the semigroup solution of (3.1), in $L^1(\mathbb{R}^N)$ uniformly on compact subsets of $[0,\infty)$. Since $\|v_{\lambda}\|_{L^{\infty}} \le \|u_0\|_{L^{\infty}}$, $v_{\lambda} \to u$ in $L^p(\mathbb{R}^N)$ for $1 \le p < \infty$, uniformly on compact sets. Thus $f_{\lambda}(t) = \frac{1}{\beta+1} \int_{\mathbb{R}^N} v_i^{\beta+1}$ for $i\lambda \le t < (i+1)\lambda$ converges uniformly

on compacts to $f(t) = \frac{1}{\beta+1} \int_{\mathbb{R}^N} (u(t))^{\beta+1}$. The relation (3.10) can be rephrased as

$$\frac{1}{\lambda} [f_\lambda(t+\lambda) - f_\lambda(t)] + C f_\lambda(t+\lambda)^{2/2^*} \le 0$$

for $0 \le t$. Multiply this relation by $\psi \in C_0^\infty((0,\infty))$ $\psi \ge 0$ and integrate to find

$$\frac{1}{\lambda} \int_0^\infty (f_\lambda(t+\lambda) - f_\lambda(t))\psi(t)dt + c \int_0^\infty f_\lambda(t+\lambda)^{2/2^*} \psi(t)dt$$

$$= \int_0^\infty f_\lambda(t)\left(\frac{\psi(t-\lambda) - \psi(t)}{\lambda}\right) dt + c\int_0^\infty f_\lambda(t+\lambda)^{2/2^*} \psi(t)dt \le 0$$

provided $\psi(t) = 0$ for $0 \le t \le \lambda$. Letting $\lambda \downarrow 0$ we have

$$-\int_0^\infty f(t)\psi'(t)dt + c \int_0^\infty f(t)^{2/2^*} \psi(t)dt \le 0$$

so $f' + c f(t)^{2/2^*} \le 0$ in $\mathcal{D}'((0,\infty))$. However, the comparison argument made above is valid if (3.7) only holds in the sense of distributions. We conclude that $f(t) \equiv 0$ for $t > T^*$, where

$$T^* = \left(\frac{1}{\beta+1} \int_{\mathbb{R}^N} u_0^{\beta+1}\right)^{1-\gamma} ((1-\gamma)c)^{-1} .$$

Now we can eliminate the assumption $u_0 \in L^\infty(\mathbb{R}^N)$ by approximation (since T^* depends only on $\|u_0\|_{L^{\beta+1}(\mathbb{R}^N)}$) .

Remarks: The solution u of the problem (which can be generalized)

$$(3.11) \qquad \begin{cases} u_t - \Delta(|u|^m \operatorname{sign} u) = 0 & t > 0 , \quad x \in \Omega \\ u(0,x) = u_0(x) & x \in \Omega \\ u^m(t,x) = 0 & \text{for } t > 0 \quad x \in \partial\Omega \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$ and $u_0 \in L^\infty(\Omega)$ has a finite extinction time T^* such that $u \equiv 0$ for $t \ge T^*$ if $0 < m < 1$. This may be proved for our notion of solution much as above. The behavior of u as $t \uparrow T^*$ is considered in Berryman and Holland [6], who remark in passing on adapting a proof of Sabinina [15] of the existence of T^* if $N = 1$. If $\Omega = \mathbb{R}^N$, as in our case, then T^* exists only if $0 < m < (N-2)/N$ as our results prove. (There can be no extinction in the case we have continuous dependence in $L^1(\mathbb{R}^N)$.) The results of Benilan and Aronson [1] establish that there is no extinction for $(N-2)/N < m \le 1$. Veron [17] exhibits other cases of extinction times.

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The initial-value problem for equations of the form $u_t - \Delta\varphi(u) = 0$ where $\varphi: \mathbb{R} \to \mathbb{R}$ is nondecreasing arises in many contexts. The main results of this paper concern the continuity of the solutions of this initial-value problem as a function of φ. psi Depending on the behavior of φ near zero, one finds either that the solutions are continuous into $C([0,\infty):L^1(\mathbb{R}^N))$ as a function of φ or into a weaker space in which $L^1(\mathbb{R}^N)$ is replaced by a certain weighted L^1 space. A variety of auxiliary results are proved and the sharpness of the condition which distinguishes between the above cases is established.

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